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Constructive Approximations to Densities Invariant under Nonexpanding Transformations

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Let τ: $[0, 1] \to [0, 1]$ be a piecewise monotonic, nonexpanding map which has an invariant density g and is topologically conjugate to a piecewise monotonic, expanding map, where the conjugacy is absolutely continuous. An effective, computable method is presented for approximating g.

KEY WORDS: Piecewise monotonic, nonexpanding transformation; absolutely continuous invariant measure; approximation of the density function.

1. INTRODUCTION

The transformation τ: $[0, 1] \to [0, 1]$ is called piecewise expanding if there exist $0 = a_0 < a_1 < \cdots < a_N = 1$ and a constant $\lambda > 1$ such that for any $i = 0, 1,..., N-1$ the following hold:

(i) $\tau|_{(a_i, a_{i+1})}$ is of class C^1 and the limits $\tau'(a_i^+)$, $\tau'(a_{i-1}^-)$ exist (or are infinite).

(ii) $|\tau'(x)| \geqslant \lambda > 1$ for $x \in (a_i, a_{i+1})$.

(iii) $|1/\tau'|$ is a function of bounded variation.

The following theorem is proved in Ref. 1.

Theorem 1. Let $\{\tau_\alpha\}_{\alpha \in \mathscr{A}}$ be a family of piecewise expanding transformations satisfying the following conditions:

(i) There exists a constant $\lambda > 1$ such that $|\tau_\alpha'(x)| \geqslant \lambda$.

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(ii) There exists a constant $W > 1$ such that for any $a \in \mathscr{A}$,

$$\text{Var}\, |1/\tau'_\alpha| \leqslant W$$

(iii) There exists a constant $\delta > 1$ such that for any $\alpha \in \mathscr{A}$, there exists a finite partition $\mathscr{K}_\alpha$ of $[0, 1]$ such that for $I \in \mathscr{K}_\alpha$, $\tau_\alpha|_I$ is one-to-one, $\tau_\alpha(I)$ is an interval, and $\min_{I \in \mathscr{K}_\alpha} \text{diam}(I) > \delta$.

(iv) For any $m \geqslant 1$, there exists $\delta_m > 0$ such that if

$$\mathscr{K}_\alpha^{(m)} = \bigvee_{j=0}^{m-1} \tau^{-j}(\mathscr{K}_\alpha)$$

then

$$\min_{I \in \mathscr{K}_\alpha^{(m)}} \text{diam}(I) \geqslant \delta_m > 0$$

Then for any density f of bounded variation, there exists a constant V such that for any $\alpha \in \mathscr{A}$ and any $k = 1, 2,...,$

$$\text{Var}\, P_{\tau_\alpha}^k f \leqslant V$$

where P_{τ_α} is the Frobenius–Perron operator associated with τ_α.

A piecewise monotonic transformation τ is called Markov with respect to the partition $\mathscr{I}$ if it transforms the set Q of endpoints of intervals of $\mathscr{I}$ into itself, i.e., $\tau(Q) \subset Q$. This condition implies that if $\text{int}(\tau(I_i) \cap I_j) \neq \varnothing$, then $\tau(I_i) \supset I_j$ for $I_i, I_j \in \mathscr{I}$. The compactness result of Theorem 1 allows the following approximation theorem.

Theorem 2.[1] Let τ be a piecewise expanding transformation. Let $\{\tau_n\}_{n \geqslant 1}$ be a family of piecewise-linear Markov approximations of τ. Then any τ_n, $n = 1, 2,...,$ admits an invariant density f_n, the set $\{f_n\}_{n \geqslant 1}$ is precompact in $\mathscr{L}_1$, and any of its limit points is an invariant density of τ.

The main objective of this paper is to prove a result similar to Theorem 2 for certain nonexpanding transformations. We shall be concerned with nonexpanding maps that have an absolutely continuous invariant measure such as the Misiurewicz maps.[2] Such maps are topologically conjugate to expanding maps[3] and the conjugacy is an absolutely continuous function. If the invariant density of the expanding map is known exactly, then, of course, the conjugacy can be used to determine the invariant density of the nonexpanding transformation. Here we are interested in the question: are densities close to the invariant density of the expanding map transformed to densities close to the invariant density of the nonexpanding map?

In Section 2, we review one of the approximation methods presented in Ref. 1. In Section 3, we show that the piecewise linear Markov maps approximating the expanding map are transformed under the conjugacy to Markov maps (not necessarily piecewise linear) that approximate the nonexpanding map and whose invariant densities converge to the invariant density of the nonexpanding map.

Since the approximating maps to the nonexpanding maps are not piecewise linear, it is in general difficult to find their invariant densities. In Section 4, we present a method that allows the direct computation of densities that approximate the invariant density of the nonexpanding map. An example is worked out in Section 5, and in Section 6, the approximation of Liapunov exponents is discussed.

2. PIECEWISE-LINEAR MARKOV APPROXIMATION FOR PIECEWISE EXPANDING TRANSFORMATIONS

Let $\tau: [0, 1] \to [0, 1]$ be piecewise expanding. Let $Q = \{0 = a_0, a_1, ..., a_N = 1\}$ and let $\mathscr{I}$ be the partition of $[0, 1]$ into closed intervals with endpoints belonging to Q, where $I_1 = [a_0, a_1], ..., I_N = [a_{N-1}, a_N]$ and $\tau|_{(a_i, a_{i+1})}$ is of class C^1.

Let $Q^{(0)} = Q$, $\mathscr{I}^{(0)} = \mathscr{I}$. We define

$$Q^{(k)} = \bigcup_{j=0}^{k} \tau^{-j}(Q^{(0)}), \qquad k = 1, 2, ...$$

$$\mathscr{I}^{(k)} = \bigvee_{j=0}^{k} \tau^{-j}(\mathscr{I}^{(0)}), \qquad k = 1, 2, ...$$

It is easy to see that $Q^{(k)}$ is the set of endpoints of intervals belonging to $\mathscr{I}^{(k)}$.

We now define a sequence of piecewise expanding Markov transformations τ_n (with respect to $\mathscr{I}^{(n)}$, $n = 1, 2, ...$), associated with τ, as follows:

(a) If $I = [a, b] \in \mathscr{I}^{(n)}$ and $I \cap Q^{(0)} = \varnothing$, then $\tau|_I$ is a piecewise linear function such that $\tau_n(a) = \tau(a)$, $\tau_n(b) = \tau(b)$.

(b) If $I = [a_i, q] \in \mathscr{I}^{(n)}$, $a_i \in Q^{(0)}$, and $\tau|_I$ is increasing, we take $q_{a_i}^{(n)} \in Q^{(n)}$ such that $q_{a_i}^{(n)} \leqslant \tau(a_i)$ and $(q_{a_i}^{(n)}, \tau(a_i)) \cap Q^{(n)} = \varnothing$. If $\tau|_I$ is decreasing, we take the point $q_{a_i}^{(n)} \in Q^{(n)}$ such that $q_{a_i}^{(n)} \geqslant \tau(a_i)$ and $(\tau(a_i), q_{a_i}^{(n)}) \cap Q^{(n)} = \varnothing$. We define $\tau_n|_I$ as a linear function such that $\tau_n(a_i) = q_{a_i}^{(n)}$, $\tau_n(q) = \tau(q)$.

(c) If $I = [q, a_i]$, $a_i \in Q^{(0)}$, the definition of $\tau_n|_I$ is analogous to that given in (b).

It is easy to see that τ_n is a piecewise linear, expanding Markov transformation with respect to the partition $\mathscr{I}^{(n)}$. Let f_n be an invariant density of τ_n. By Theorem 2, we have $f_n \to f$, the invariant density of τ, in $\mathscr{L}_1$. The importance of choosing piecewise linear Markov maps lies in the fact that invariant densities are step functions which are the left eigenvectors of simple matrices.[4]

The foregoing procedure can be written in the form of an algorithm:

Step 1. Let $\mathscr{I}^{(0)} = \{I_i\}_{i=0}^{N-1}$ be the partition of I into the intervals of smoothness of τ. Set $\mathscr{I}^{(k)} = \bigvee_{i=0}^{k} \tau^{-i}(\mathscr{I})$. Let

$$Q^{(k)} = \{q_1^{(k)},..., q_{r_k}^{(k)}\}$$

be the set of endpoints of the intervals that are elements of the partition $\mathscr{I}^{(k)}$.

Step 2. Form the piecewise linear Markov map τ_k on the partition $\mathscr{I}^{(k)}$ by choosing the images of the original points $\{a_j^-, a_j^+\}$ in such a way that τ_k has the magnitude of its slope greater than or equal to that of τ everywhere that it is defined.

Step 3. Let the matrix M_k denote the Frobenius–Perron operator of P_{τ_k}, restricted to the space of step functions on $\mathscr{I}^{(k)}$. Compute the left eigenvector of M_k,[4] f_k, which we view as a step functions on $\mathscr{I}^{(k)}$. (By Theorem 2, f_k approximates f.)

For $Q^{(k)}$, each interval of the associated partition $\mathscr{I}^{(k)}$ corresponds to one row of the matrix M_k. Recall,

$$\begin{aligned} m_{ij} &= \frac{1}{|\tau'_k|_{I_i^{(k)}}|} \qquad \text{if} \quad \tau_k(I_i^{(k)}) \supseteq I_j^{(k)} \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

where $I_i^{(k)} \in \mathscr{I}^{(k)}$. In computing $\tau_k|_{I_i^{(k)}}$, we choose the closest partition points of $Q^{(k)}$ which produces a slope for τ_k that is larger in magnitude than that of $\tau|_{I_i^{(k)}}$.

Gaussian elimination is used to find the left eigenvector of M_k, which is then normalized by the requirement that

$$\sum_{\{i:\, I_i^{(k)} \in \mathscr{I}^{(k)}\}} f_k|_{I_i^{(k)}}\, m(I_i^{(k)}) = 1$$

where m is Lebesgue measure on $[0, 1]$.

3. NONEXPANDING MAPS

The method of approximating an invariant density described in Section 2 requires the transformation to be expanding. The question we address in this section is whether an analogous procedure can be applied when the transformation τ is not expanding. In general it is not true that the invariant densities of Markov transformation approximations converge to the invariant density of the limit transformation. For $\tau(x)=4x(1-x)$, it can be shown[5] that there exists a sequence of Markov maps $\{\tau_n\}$ approaching τ uniformly but such that the corresponding invariant densities $\{f_n\}$ approach the point measure at 0.

In the sequel we shall prove that Markov transformations approximating τ can be found with the property that the associated invariant densities converge weakly to f, the invariant density of τ. The assumption we make is that τ is topologically conjugate to an expanding map via an absolutely continuous homeomorphism. In Ref. 3 it is shown that this is true for a large class of nonexpanding maps that includes the Misiurewicz maps.[2]

We shall need the following preliminary result.

Lemma 1. Let $\tau_n \to \tau$ uniformly, $P_{\tau_n} f_n = f_n$, and $f_n \to f$ weakly in $\mathscr{L}_1$. Then $P_\tau f = f$.

Proof. We shall prove that measures $f\, dm$ and $(P_\tau f)\, dm$ are equal. It is enough to prove that for any $g \in C(X)$

$$\int g(f - P_\tau f)\, dm = 0$$

We have

$$\left|\int g(f-P_\tau f)\, dm\right| \leqslant \left|\int g(f-f_n)\, dm\right| + \left|\int g(f_n - P_n f_n)\, dm\right| + \left|\int g(P_n f_n - P_\tau f_n)\, dm\right| + \left|\int g(P_\tau f_n - P_\tau f)\, dm\right|$$

where $P_n = P_{\tau_n}$. The first summand tends to 0, since $f_n \to f$ weakly. The second is equal to 0. The fourth is equal to $|\int (g\circ\tau)(f_n - f)\, dm|$ and goes to zero, since $f_n \to f$ weakly (if g is continuous, then $g\circ\tau$ is bounded). The third summand is equal to

$$\left|\int (g\circ\tau_n - g\circ\tau) f_n\, dm\right| \leqslant \sup_x |g\circ\tau_n(x) - g\circ\tau(x)| \int |f_n|\, dm$$

$$\leqslant \omega_g(\sup_x |\tau_n(x) - \tau(x)|) \int |f_n|\, dm$$

where ω_g is a modulus of continuity of g. By assumption, $\sup_x |\tau_n(x) - \tau(x)|$ tends to 0, so $\omega_g(\sup_x |\tau_n(x) - \tau(x)|) \to 0$. The integrals $\int |f_n| \, dm$ are uniformly bounded, since $\{f_n\}$ is a weakly compact set in $\mathscr{L}_1$. Thus, the lemma is proved. ▮

Lemma 2. Let $\tau: I \to I$ be a nonexpanding map topologically conjugate to T, i.e., $\tau = h^{-1} \circ T \circ h$, where h is absolutely continuous and T, a piecewise expanding transformation, admits a unique, absolutely continuous, invariant measure μ with density f. Then there exists a sequence of maps $\{\tau_n\}$ such that $\tau_n \to \tau$ uniformly, and such that the set of densities $\{g_n\}$ corresponding to $\{\tau_n\}$ is weakly compact in $\mathscr{L}_1$.

Proof. Let $Q^{(n)}$ be the set of endpoints corresponding of the partition $\mathscr{I}^{(n)} = \bigvee_{i=0}^{n} T^{-i}(I)$, where $I = \{I_0, I_1, ..., I_{N-1}\}$ are the subintervals of the partition $\mathscr{I}$ defined by $0 = a_0 < a_1 < \cdots < a_{N-1} < a_n = 1$, such that T is C^2 on (a_i, a_{i+1}), $i = 0, ..., n$.

Let T_n be the piecewise linear Markov approximation to T on $\mathscr{I}^{(n)}$. Define τ_n by

$$\tau_n \circ h^{-1} = h^{-1} \circ T_n$$

Let $x_i \in Q^{(n)}$. Then $\tau_n(h^{-1}(x_i)) = h^{-1}(T_n(x_i))$. Since T_n is Markov on $Q^{(n)}$, $T_n(x_i) \in Q^{(n)}$. Hence, $\tau_n(h^{-1}(x_i)) \subset h^{-1}(Q^{(n)})$, and $\tau_n(h^{-1}(Q^{(n)})) \subset h^{-1}(Q^{(n)})$. Thus, τ_n is a Markov map on $h^{-1}(Q^{(n)})$. Note that, in general, τ_n is not piecewise linear.

We claim that $\tau_n \to \tau$ uniformly. Let ω denote the modulus of continuity of h^{-1}. Then

$$\begin{aligned} \sup_y |\tau_n(y) - \tau(y)| &= \sup_x |\tau_n(h^{-1}(x)) - \tau(h^{-1}(x))| \\ &= \sup_x |h^{-1}(T_n(x)) - h^{-1}(T(x))| \\ &\leqslant \omega(\sup_x (T_n(x) - T(x))) \end{aligned}$$

Since $T_n(x) \to T(x)$ uniformly and h^{-1} is uniformly continuous on $[0, 1]$,

$$\sup_y |\tau_n(y) - \tau(y)| \to 0 \quad \text{as} \quad n \to \infty$$

Let f_n be the density corresponding to T_n. By Theorem 2, $\{f_n\}$ is compact in $\mathscr{L}_1$. By the conjugacy relation,

$$g_n(x) = f_n(h(x)) \frac{dh}{dx}$$

By Ref. 3, h is absolutely continuous, so the derivative exists. We claim that $\{g_n\}$ is weakly compact in $\mathscr{L}_1$. By the change-of-variable theorem,

$$\int_A g_n(x)\,dx = \int_A f_n(h(x))\frac{dh}{dx}\,dx = \int_{h(A)} f_n(y)\,dy$$

We want to show that for each $\varepsilon > 0$, $\exists\delta > 0$ such that $m(A) < \delta$ implies $\int_A g_n(x)\,dx < \varepsilon$ for all n. Since $\{f_n\}$ is weakly compact in $\mathscr{L}_1$, given $\varepsilon > 0$, $\exists\delta_1 > 0$ such that $l(h(A)) < \delta_1$ implies

$$\int_{h(A)} f_n(y)\,dy < \varepsilon$$

for all n. But h is absolutely continuous. Thus, given $\delta_1 > 0$, there exists $\delta > 0$ such that $m(A) < \delta$ implies $m(h(A)) < \delta_1$. Hence, we have

$$\int_A g_n(x)\,dx < \varepsilon$$

for all n if $m(A) < \delta$. Thus, $\{g_n\}$ is weakly compact in $\mathscr{L}_1$. ∎

Theorem 3. $g_n \to g$, where g is the invariant density of τ.

Proof. Since $\tau_n \to \tau$ uniformly, Lemma 1 yields the desired result. ∎

4. CONSTRUCTIVE APPROXIMATIONS TO THE INVARIANT DENSITY OF NONEXPANDING TRANSFORMATIONS

Let τ: $[0, 1] \to [0, 1]$ be a piecewise C^1, nonexpanding transformation having an invariant density g. As in the previous section, we assume there exists an absolutely continuous homeomorphism h that conjugates τ with a piecewise expanding transformation T, i.e., $T = h \circ \tau \circ h^{-1}$.

We need a few preliminary lemmas. Let C_S denote the set of turning points of the transformation S: $[0, 1] \to [0, 1]$ together with the points $\{0, 1\}$.

Lemma 3. $C_T = h(C_\tau)$.

Proof. For almost every $x \in [0, 1]$,

$$T'(x) = h'(\tau \circ h^{-1}(x)) \cdot \tau'(h^{-1}(x)) \cdot (h^{-1})'(x)$$

Since $h'(h^{-1})'$ is always positive, the signs of $T'(x)$ and $\tau'(h^{-1}(x))$ are the same. ∎

Lemma 4. Let $Q_\tau^{(n)} = \bigcup_{k=0}^{n} \tau^{-k}(C_\tau)$ and $Q_T^{(n)} = \bigcup_{k=0}^{n} T^{-k}(C_T)$. Then $Q_T^{(n)} = h(Q_\tau^{(n)})$.

Proof. Let $x \in h(Q^{(n)})$, i.e., $x = h(\tau^{-k}c)$, $0 \leqslant k \leqslant n$, $c \in C_\tau$. But $h^{-1}T^{-k}h(c) = \tau^{-k}(c)$. Thus, $x = T^{-k}(h(c))$, which implies that $x \in Q_T^{(n)}$ and so $h(Q_\tau^{(n)}) \subset Q_T^{(n)}$. The reverse inclusion can be proved analogously.

Lemma 5. Let $\mathscr{B}_\tau = \bigcup_{n \geqslant 0} Q_\tau^{(n)}$, $\mathscr{B}_T = \bigcup_{n \geqslant 0} Q_T^{(n)}$. Then we have $\mathscr{B}_T = h(\mathscr{B}_\tau)$ and $\overline{\mathscr{B}_\tau} = \overline{\mathscr{B}_T} = [0, 1]$.

Proof. The equality $\mathscr{B}_T = h(\mathscr{B}_\tau)$ follows directly from Lemma 4. Since T is a piecewise expanding transformation, $\overline{\mathscr{B}_T} = [0, 1]$. But h^{-1} is also a homeomorphism, so $\mathscr{B}_\tau = h^{-1}(\mathscr{B}_T)$ is also dense in $[0, 1]$.

Let $I_\tau^{(n)}$ and $I_T^{(n)}$ be partitions of $[0, 1]$ into intervals with endpoints in $Q_\tau^{(n)}$ and $Q_T^{(n)}$, respectively. Let T_n be a piecewise linear Markov (on the partition $I_T^{(n)}$) approximation to T, as described in Section 2. We know, by Theorem 2, that the invariant densities f_n of T_n converge in $\mathscr{L}_1$ to f, the invariant density of T.

Now let h_n be a piecewise linear approximation to h on the partition $I_\tau^{(n)}$. If we define τ_n by $\tau_n = h_{n-1}^{-1} \circ T_n \circ h_n$, τ_n would be a piecewise linear and Markov (on the partition $I_\tau^{(n)}$) approximation to τ. Unfortunately, such τ_n are not conjugate to T_n. We therefore define τ_n by $\tau_n = h_n^{-1} \circ T_n \circ h_n$. The transformation τ_n is piecewise linear (but not with respect to $I_\tau^{(n)}$), and it approximates τ uniformly.

We will now prove that every τ_n has an invariant density and that $\{g_n\}$ forms a weakly compact set in $\mathscr{L}_1$. By Lemma 1, this implies that every weak limit point of $\{g_n\}$ is an invariant density of τ.

Proposition 1. The density $g_n(x) = f_n(h_n(x))\, h_n'(x)$ is invariant under τ_n, $n = 1, 2, \ldots$; the set $\{g_n\}$ is weakly compact in $\mathscr{L}_1$, and every weak limit point of $\{g_n\}$ is an invariant density of τ.

Proof. It is easy to check that g_n is τ_n-invariant. To prove the compactness of $\{g_n\}_{n \geqslant 1}$, it is enough to show that for any $\varepsilon > 0$, $\exists \delta > 0$ such that for any measurable set A, $m(A) < \delta$,

$$\int_A g_n(x)\, dx < \varepsilon, \qquad \text{i.e.,} \qquad \int_{h_n(A)} f_n(x)\, dx < \varepsilon \tag{1}$$

Since $\{f_n\}_{n \geqslant 1}$ is weakly compact in $\mathscr{L}_1$, to prove (1) it is enough to prove that for any $\eta > 0$, $\exists \rho > 0$ such that for any measurable set A, $m(A) < \rho$, and for all n,

$$m(h_n(A)) < \eta \tag{2}$$

Since h_n is absolutely continuous,

$$m(h_n(A)) = \int_A h'_n(x)\,dx$$

The set $\{h'_n\}$ is strongly compact in $\mathscr{L}_1$, since $h'_n \to h'$ in $\mathscr{L}_1$ (h_n is the piecewise linear approximation to h with respect to $I_i^{(n)}$, and $\overline{\mathscr{B}_\tau} = [0, 1]$). This implies (2). Invoking Lemma 1 completes the proof.

Note that the density g_n is constant on elements of $I_\tau^{(n)}$ and it can be effectively computed. We have

$$g_n = f_n(h_n(x))\,h'_n(x) = f_n(h_n(x))\,\frac{m(h(J))}{m(J)} \tag{3}$$

where $x \in J \in I_\tau^{(n)}$.

Corollary 1. Let τ: $[0, 1] \to [0, 1]$ be a nonexpanding transformation such that some iterate τ^n is topological conjugate to a transformation T, where T^m is piecewise expanding and the conjugacy is absolutely continuous. Then the approximation methods of Proposition 1 can be used for τ.

Proof. We are given that $\tau^n = h^{-1} \circ T \circ h$ and T^m is piecewise expanding. Define $S = h \circ \tau \circ h^{-1}$. Then $\tau = h^{-1} \circ S \circ h$ and $\tau^n = h^{-1} \circ S^n \circ h$. Thus, $T = S^n$. Since $T^m = S^{n \cdot m}$ is piecewise expanding, Corollary 1 of Ref. 1 yields the desired results.

We now summarize the method for obtaining the g_n in the form of an algorithm. We start with a given nonexpanding transformation τ and the piecewise expanding transformation T which is absolutely continuously conjugated to τ.

Algorithm 1

1. Compute $Q_T^{(n)}$ and the piecewise linear Markov approximation T_n [at every step only $T^{-n}(C_T)$ is calculated].
2. Compute $Q_\tau^{(n)}$ [at every step only $\tau^{-n}(C_\tau)$ is calculated].
3. Compute the piecewise constant, T_n-invariant density f_n using the matrix representation for the Frobenius–Perron operator of T_n.[4]
4. Compute g_n using (3).

As a simple example of this algorithm, consider $\tau(x) = 4x(1-x)$,

$$T(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

where the conjugacy $h^{-1}(x)=\sin^2(\pi x/2)$ is obviously absolutely continuous. Now,

$$Q_T^{(n)}=\left\{\frac{k}{2^n}\right\}_{k=0}^{2^n}, \qquad Q_\tau^{(n)}=\left\{\sin^2\left(\frac{\pi}{2}\frac{k}{2^n}\right)\right\}_{k=0}^{2^n}$$

It is easy to show that f_n is constant and equal to 1 on every element of $I_T^{(n)}$. Let

$$J_k=\left[\frac{k}{2^n},\frac{k+1}{2^n}\right]$$

We obtain

$$\begin{aligned}
g_n|_{J_k}(x)&=\frac{(k+1)/2^n-k/2^n}{\sin^2[\frac{1}{2}\pi(k+1)/2^n]-\sin^2(\frac{1}{2}\pi k/2^n)}\\
&=\frac{1}{2^n\{\sin^2[\frac{1}{2}\pi(k+1)/2^n]-\sin^2(\frac{1}{2}\pi k/2^n)\}}\\
&=\frac{1}{2^n(\pi/2)2\sin(\frac{1}{2}\pi\tilde{x})\cos[\frac{1}{2}\pi\tilde{x}(1/2^n)]}, \qquad \tilde{x}\in J_k\\
&\to\frac{1}{\pi\sin(\arcsin\sqrt{x})\cos(\arccos\sqrt{x})}\\
&=\frac{1}{\pi[x(1-x)]^{1/2}}
\end{aligned}$$

as $n\to\infty$, since $\tilde{x}\to(2/\pi)\arcsin\sqrt{x}$ as $n\to\infty$.

Corollary 2. Let $\tau_n=h_{n-1}^{-1}\circ T_n\circ h_n$ and let ν_n be defined by $\nu_n(E)=\mu_n(h_n(E))$, where μ_n is a measure invariant under T_n having density f_n. Then ν_n has density g_n, as above, and hence the set $\{g_n\}_{n\geq 1}$ is weakly compact. The (unique) limit point g is the invariant density of τ.

Proof. By definition,

$$\nu_n(E)=\int_E g_n\,dx=\int_{h_n^{-1}(E)} f_n\,dx$$

Since

$$\int_{h_n^{-1}(E)} f_n\,dx=\int_E (f_n\circ h_n)\,h_n'\,dx \quad \text{a.e.}$$

we have $g_n=(f_n\circ h_n)\,h_n'$ a.e., as above, and the result follows. Note that

$$\begin{aligned}
\nu_n(\tau_n^{-1}E)&=\mu_n(h_n\tau_n^{-1}E)=\mu_n(T_n^{-1}h_{n-1}E)\\
&=\mu_n(h_{n-1}E)\neq\mu_n(h_nE)
\end{aligned}$$

in general, so ν_n is not τ_n-invariant. The advantage of this approach is that τ_n is piecewise linear and Markov with respect to the partition $I_\tau^{(n)}$, and hence the densities g_n induced by f_n must be piecewise constant with respect to the partition $I_\tau^{(n)}$, since the f_n are piecewise constant with respect to $I_T^{(n)}$. Moreover, the densities g_n are "approximately invariant" for large n in the sense that

$$|\nu_n(E_n) - \nu(\tau_n^{-1} E_n)| = |\mu_n(h_n E_n) - \mu(h_{n-1} E_n)| \to 0$$

as $n \to \infty$. This follows from the fact that $\{g_n\}$ is a (weak) Cauchy sequence in $\mathscr{L}_1$, as was shown above.

Example. Let us again consider

$$T(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

and $\tau(x) = 4x(1-x)$. Then $\tau(x) = h^{-1} \circ T \circ h(x)$, where $h(x) = (2/\pi) \arcsin(\sqrt{x})$, $h^{-1}(x) = \sin^2(\pi x/2)$, and $I_\tau^{(n)} = \bigvee_{i=0}^{n} \tau^{-1}(\{(0, \frac{1}{2}), (\frac{1}{2}, 1)\})$, $Q_T^{(n)} = \{j/2^{n+1} : 0 \leqslant j \leqslant 2^{n+1}\}$, and let $I_T^{(n)}$ be the subintervals with endpoints from $Q_T^{(n)}$. Then $I_\tau^{(n)} = h_n(I_T^{(n)})$. Since $f_n = 1$ on $[0, 1]$, we have

$$\begin{aligned} g_n &= (f_n \circ h_n)\, h_n' = h_n' \\ &= \frac{1}{(h_n^{-1})'(h_n(x))} \end{aligned}$$

and

$$\begin{aligned} h_n^{-1}(x)\,|_{[j/2^{n+1},\,(j+1)/2^{n+1}]} &= \sin^2\left(\frac{\pi}{2}\frac{j}{2^{n+1}}\right) \\ &\quad + \left[\sin^2\left(\frac{\pi}{2}\frac{j+1}{2^{n+1}}\right) - \sin^2\left(\frac{\pi}{2}\frac{j}{2^{n+1}}\right)\right](2^{n+1}x - j) \end{aligned}$$

Thus, on $[j/2^{n+1}, (j+1)/2^{n+1}]$,

$$\begin{aligned} \frac{1}{(h_n^{-1})'} &= \frac{1}{2^{n+1}} \frac{1}{\sin^2[\frac{1}{2}\pi(j+1)/2^{n+1}] - \sin^2(\frac{1}{2}\pi j/2^{n+1})} \\ &= \frac{1}{2^{n+1}} \frac{1}{\sin[\frac{1}{2}\pi(2j+1)/2^{n+1}]\, \sin[\frac{1}{2}\pi(1/2^{n+1})]} \\ &= \frac{2}{\pi} \frac{1}{2\sin[\frac{1}{2}\pi(2j+1)/2^{n+2}]\, \cos[\frac{1}{2}\pi(2j+1)/2^{n+2}]} \\ &\quad \times \frac{\frac{1}{2}\pi(1/2^{n+1})}{\sin[\frac{1}{2}\pi(1/2^{n+1})]} \end{aligned}$$

Therefore, for

$$x \in \left[h_n^{-1}\left(\frac{j}{2^{n+1}}\right), h_n^{-1}\left(\frac{j+1}{2^{n+1}}\right) \right]$$

we have $g_n(x) = g_n(x_0)$, where

$$x_0 = h_n^{-1}\left(\frac{2j+1}{2^{n+1}}\right)$$

i.e.,

$$g_n(x) = \frac{1}{\pi} \frac{1+\varepsilon_n}{\sin[\frac{1}{2}\pi h_n(x_0)] \cos[\frac{1}{2}\pi h_n(x_0)]}$$

$$= \frac{1}{\pi} \frac{1+\varepsilon_n}{[x_0(1-x_0)]^{1/2}}$$

where

$$1+\varepsilon_n = \frac{\frac{1}{2}\pi(1/2^{n+1})}{\sin[\frac{1}{2}\pi(1/2^{n+1})]}$$

i.e., g_n converges strongly in $\mathscr{L}_1$ to $g(x) = 1/\{\pi[x(1-x)]^{1/2}\}$.

5. AN EXAMPLE

Consider the piecewise expanding map

$$T(x) = \begin{cases} 2x + \frac{1}{2}, & 0 \leqslant x \leqslant \frac{1}{4} \\ \frac{14}{45}x^2 - \frac{31}{18}x + \frac{127}{90}, & \frac{1}{4} \leqslant x \leqslant 1 \end{cases}$$

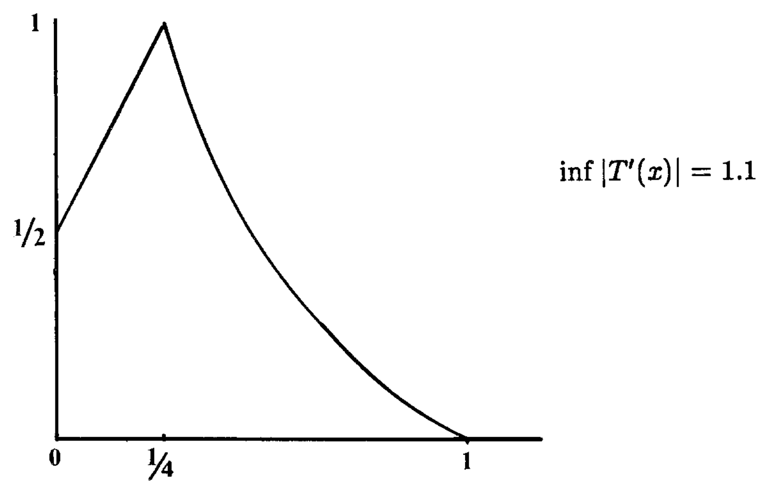


Fig. 1. An expanding map.

shown in Fig. 1. Define $\tau(x)=h^{-1}\circ T\circ h(x)$, where $h(x)=x^2$. Then

$$\tau(x)=\begin{cases}(2x^2+\frac{1}{2})^{1/2}, & 0\leqslant x\leqslant\frac{1}{2}\\ (\frac{14}{45}x^4-\frac{31}{18}x^2+\frac{127}{90})^{1/2}, & \frac{1}{2}\leqslant x\leqslant 1\end{cases}$$

Since in a small interval $[0,\alpha]$,

$$T^n(x)=P^n(x)+d_n$$

where $P^n(x)$ is a polynomial with no constant term and d_n is a constant $\neq 0$, then

$$\tau^n(x)=[P^n(x^2)+d_n]^{1/2}, \qquad x^2\in[0,\alpha]$$

and

$$(\tau^n)'(x)=\frac{(P^n)'(x^2)\cdot x}{[P^n(x^2)+d_n]^{1/2}}, \qquad x^2\in[0,\alpha]$$

which can be made arbitrarily small in absolute value for x near 0, since $d_n\neq 0$. Hence, every iterate of τ has a region with slope <1 in absolute value.

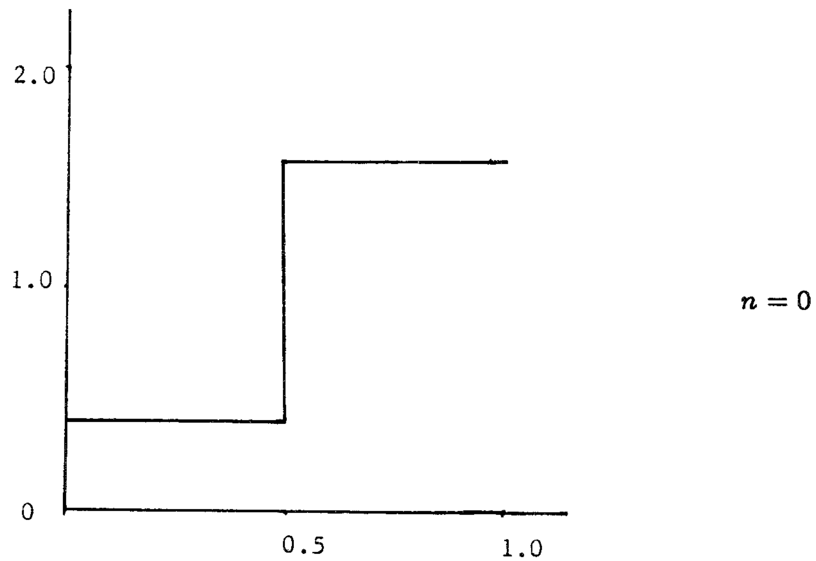
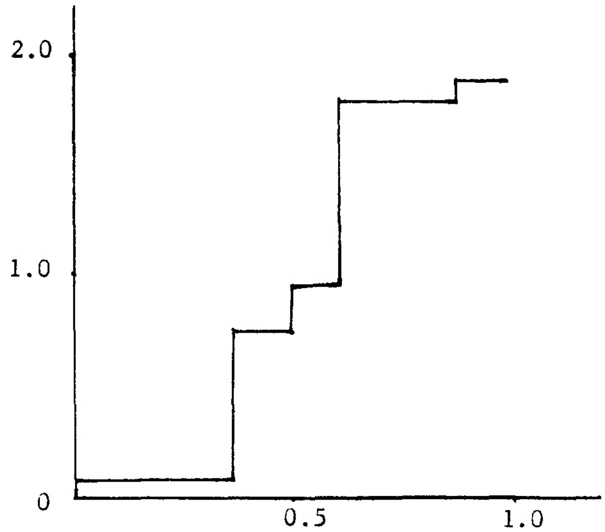
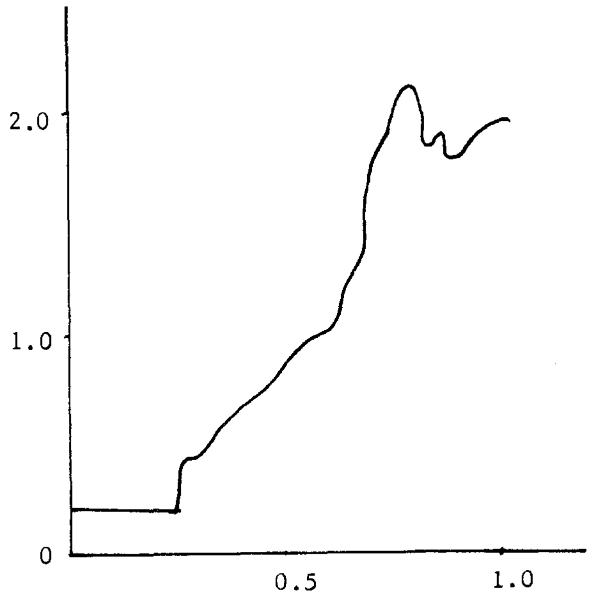


Fig. 2. Density function.



$n = 3$

Fig. 3. Density function.



$n = 6$

Fig. 4. Smoothed density function.

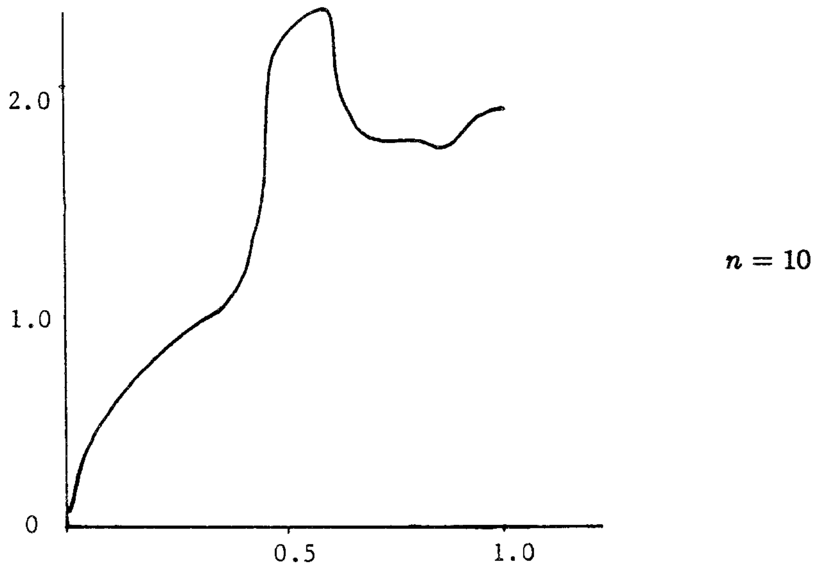


Fig. 5. Smoothed density function.

Following Algorithm 1, we compute f_n, the invariant density of T_n, the piecewise linear Markov approximation to T on the partition $I_T^{(n)} = \{0 = a_0^{(n)} < a_1^{(n)} < \cdots < a_q^{(n)} = 1\}$. Then, using (3), we obtain for $x \in J_k \in I_\tau^{(n)}$,

$$g_n(x) = f_n(h_n(x)) \frac{a_k^{(n)} - a_{k-1}^{(n)}}{(a_k^{(n)})^{1/2} - (a_{k-1}^{(n)})^{1/2}}$$

$$= f_n(h_n(x))[(a_k^{(n)})^{1/2} + (a_{k-1}^{(n)})^{1/2}]$$

Thus,

$$g_n|_{J_k} = f_n|_{h_n(J_k)} [(a_k^{(n)})^{1/2} + (a_{k-1}^{(n)})^{1/2}]$$

Graphs of g_n as a function of n are shown in Figs. 2–5.

6. APPROXIMATING THE LIAPUNOV EXPONENT OF NONEXPANDING TRANSFORMATIONS

In Ref. 6 the method outlined in Section 2 is used to approximate the Liapunov exponent of piecewise expanding transformations. For nonexpanding maps, Theorem 3 of Ref. 6 required the invariant densities of the approximating transformations to be weakly compact. In view of Proposition 1, this condition is no longer necessary if τ is topologically conjugate to a piecewise expanding transformation via a homeomorphism that is absolutely continuous.

Let $\tau = h^{-1} \circ T \circ h$ be nonexpanding and let $\tau_n = h_{n-1}^{-1} \circ T_n \circ h_n$, which is piecewise linear and Markov. The densities g_n converge weakly to g, the invariant density of g, as $n \to \infty$, by Corollary 2. Since $\tau_n' \to \tau'$ uniformly and $g_n \to g$ weakly, we have

$$\lambda_n = \int_0^1 g_n(x) \log_2 |\tau_n'(x)| \, dx$$

$$\to \int_0^1 g(x) \log_2 |\tau'(x)| \, dx$$

which is the Liapunov exponent of τ.

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